

Optimization and Decision Making 2020 Examination: Solutions

1

a) [5 marks] The variance is $\sigma_P^2 = E[(\rho_P - \mu_P)^2] = E\left[\left(\sum_{i=1}^n w_i \rho_i - \sum_{i=1}^n w_i \mu_i\right)^2\right] = E\left[\sum_{i=1}^n (w_i (\rho_i - \mu_i))^2\right] = \sum_{i,j=1}^n w_i w_j E[(\rho_i - \mu_i)(\rho_j - \mu_j)] = \sum_{i,j=1}^n w_i w_j \sigma_{ij}$.

b) [10 marks] We wish to minimise $f(w_1, \dots, w_n) = \frac{1}{2} \sum_{i,j=1}^n w_i w_j \sigma_{ij}$, subject to $g_1(w_1, \dots, w_n) = \sum_{i=1}^n w_i \mu_i - \mu_P = 0$, and $g_2(w_1, \dots, w_n) = \sum_{i=1}^n w_i - 1 = 0$.

We consider the following Lagrangian:

$L(w_1, \dots, w_n | \lambda_1, \lambda_2) = f(w_1, \dots, w_n) - \lambda_1 g_1(w_1, \dots, w_n) - \lambda_2 g_2(w_1, \dots, w_n)$. To find our optimal portfolio, we need to take all of the derivatives of the Lagrangian and set each of them to be equal to 0.

$$\begin{aligned}\frac{\partial L}{\partial w_i} &= w_i \sigma_i^2 + \sum_{j \neq i}^n w_j \sigma_{ij} - \lambda_1 \mu_i - \lambda_2 = 0, \quad i = 1, \dots, n \\ \frac{\partial L}{\partial \lambda_1} &= \mu_P - \sum_{i=1}^n w_i \mu_i = 0, \\ \frac{\partial L}{\partial \lambda_2} &= 1 - \sum_{i=1}^n w_i = 0.\end{aligned}$$

For three assets we must solve the simultaneous equations

$$\begin{aligned}\frac{\partial L}{\partial w_1} &= w_1 \sigma_1^2 + w_2 \sigma_{12} + w_3 \sigma_{13} - \lambda_1 \mu_1 - \lambda_2 = 0, \\ \frac{\partial L}{\partial w_2} &= w_2 \sigma_2^2 + w_1 \sigma_{12} + w_3 \sigma_{23} - \lambda_1 \mu_2 - \lambda_2 = 0, \\ \frac{\partial L}{\partial w_3} &= w_3 \sigma_3^2 + w_1 \sigma_{13} + w_2 \sigma_{23} - \lambda_1 \mu_3 - \lambda_2 = 0, \\ \frac{\partial L}{\partial \lambda_1} &= \mu_P - w_1 \mu_1 - w_2 \mu_2 - w_3 \mu_3 = 0, \\ \frac{\partial L}{\partial \lambda_2} &= 1 - w_1 - w_2 - w_3 = 0.\end{aligned}$$

After working (the key steps of which must be given) these rearrange to give

$$\begin{aligned}\mu_P - w_1 \mu_1 - w_2 \mu_2 - w_3 \mu_3 &= 0, \\ 1 - w_1 - w_2 - w_3 &= 0, \\ [\mu_2(\sigma_1^2 - \sigma_{13}) + \mu_1(\sigma_{13} - \sigma_{12}) + \mu_3(\sigma_{12} - \sigma_1^2)]w_1 + \\ &1\end{aligned}$$

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$$[\mu_3(\sigma_2^2 - \sigma_{12}) + \mu_2(\sigma_{12} - \sigma_{23}) + \mu_1(\sigma_{23} - \sigma_2^2)]w_2 +$$

$$[\mu_1(\sigma_3^2 - \sigma_{23}) + \mu_3(\sigma_{23} - \sigma_{13}) + \mu_2(\sigma_{13} - \sigma_3^2)]w_3 = 0.$$

The first two equations give

$$w_2 = \frac{(\mu_P - \mu_3) - w_1(\mu_1 - \mu_3)}{\mu_2 - \mu_3},$$

$$w_3 = \frac{(\mu_2 - \mu_P) + w_1(\mu_1 - \mu_2)}{\mu_2 - \mu_3}.$$

c) [10 marks] Substituting, we have $w_2 = (-0.03 + w_1 0.04)/(-0.03) = (3 - 4w_1)/3$ and

$$w_3 = (0 - 0.01w_1)/(-0.03) = w_1/3.$$

Using the third equation, we have

$$0w_1 + 0.004w_2 - 0.002w_3 = 0 \Rightarrow 4w_2 - 2w_3 = 0 \Rightarrow$$

$$w_1/6 = (3 - 4w_1)/3 \Rightarrow w_1 = 6 - 8w_1 \Rightarrow w_1 = 2/3. \text{ This in turn yields } w_2 = 1/9, w_3 = 2/9.$$

d) [5 marks] Substituting, we have $w_2 = (-0.02 + w_1 0.04)/(-0.03) = (2 - 4w_1)/3$ and

$$w_3 = (-0.01 - 0.01w_1)/(-0.03) = (1 + w_1)/3.$$

The third equation is unchanged, so we have

$$(4 - 8w_1)/3 = (1 + w_1)/3 \Rightarrow 9w_1 = 3 \Rightarrow w_1 = 1/3. \text{ This in turn yields } w_2 = 2/9, w_3 = 4/9.$$

2

a) [9 marks] **Konig's Theorem**

Let A be a square matrix. Then the maximum number of independent zeros of A is equal to the minimum number of lines required to cover all of the zeros of A .

The following method to solve the assignment problem is called the *Hungarian Method*.

Define $R = \max_{i,j} r_{ij}$, and the matrix S by
 $S = \{s_{ij} : s_{ij} = r - r_{ij}\}.$

This reduces the problem to finding n independent elements of S which give a minimum sum.

Step 1 Subtract the smallest element in each row of S from every element in that row, giving S_1 .

Step 2 Subtract the smallest element in each column of S_1 from every element in that column, giving A .

Step 3 Find k_m , the maximum number of independent zeros of A . By Konig's Theorem, k_m is also the minimum number of lines needed to cover all the zeros of A .

If $k_m = n$, then there are n independent zeros of A , and the elements in the corresponding positions of R give a solution to the original assignment problem.

Step 4 If $k_m < n$, let N denote the matrix of non-covered elements of A , and let h be the smallest element of N . Add h to each twice covered element of A , and subtract h from each non-covered element of A , denoting the new matrix as A^* .

Now repeat step 3 using A^* instead of A . If k_m is now equal to n , the problem is solved. If not treat A^* as A and thus obtain a new A and A^* . This is continued until $k_m = n$.

		Cand/Job	1	2	3	4
b) [15 marks] We have the table	A		90	80	70	85
	B		70	55	40	65
	C		75	85	45	60
	D		55	65	40	45

How should we allocated the candidates to the jobs? Finding S gives

	1	2	3	4			c			
A	0	10	20	5	. Steps 1 and 2 give	c	0	10	0	0
B	20	35	50	25		c	0	15	10	0
C	15	5	45	30			10	0	20	20
D	35	25	50	45			10	0	5	15

There is a maximum of three independent zeros. Step 4 yields N as $\begin{matrix} 10 & 20 & 20 \\ 10 & 5 & 15 \end{matrix}$

which gives $h = 1$. This then yields A^* as

	1	2	3	4
A	0	15*	0	0
B	0	20	10	0
C	5	0	15	15
D	5	0	0	10

We have four independent zeros (two possibilities). Thus the solution is given by giving

job 2 to team member C,

job 3 to team member D,

job 1 to team member A (or B),

job 4 to team member B (or A).

The sum of the values of R that we have maximised is $90 + 65 + 85 + 40 = 280$.

c) [3 marks] The new problem is simply the old problem with row B and column 3 removed, since we must allocate jobs A, C and D to candidates 1, 2 and 4.

d) [3 marks] In the new problem, replace the score for individual 3 performing job B with a large negative number, so that this is clearly not selected, then proceed as normal in the analysis.

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a) [12 marks] Maximise $80x_1 + 120x_2 + 160x_3 + 200x_4$
 subject to $2x_1 + 4x_2 + 8x_3 + 10x_4 \leq 4000$,
 $8x_1 + 12x_2 + 12x_3 + 16x_4 \leq 8000$,
 $x_1, x_2, x_3, x_4 \geq 0$.

b) [15 marks]

	x_1	x_2	x_3	x_4	
y_1	2	4	8	10	4000
y_2	8	12	12	16	8000
	-80	-120	-160	-200	0

→

	x_1	x_2	y_1	x_4	
x_3	1/4	1/2	1/8	5/4	500
y_2	5	6	-3/2	1	2000
	-40	-40	20	0	80000

→

	y_2	x_2	y_1	x_4	
x_3	-1/20				400
x_1	1/5	6/5	-3/10	1/5	400
	8	8	8	8	96000

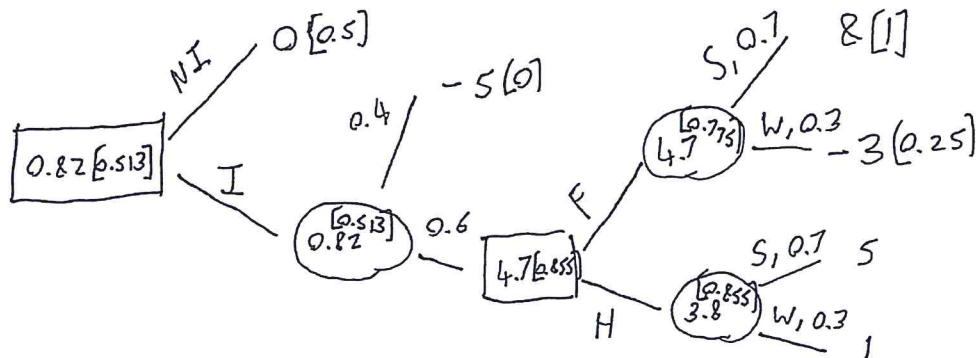
Thus the optimal solution is $x_1 = 400$ Dining Chairs and $x_3 = 400$ Reclining Chairs with profit £96000.

c) [3 marks] The new problem is the same as the problem given in a), but with four extra constraints. These are:

$$\begin{aligned} x_1 &\leq 0.4(x_1 + x_2 + x_3 + x_4) \Rightarrow 0.6x_1 - 0.4x_2 - 0.4x_3 - 0.4x_4 \leq 0, \\ x_2 &\leq 0.4(x_1 + x_2 + x_3 + x_4) \Rightarrow 0.6x_2 - 0.4x_1 - 0.4x_3 - 0.4x_4 \leq 0, \\ x_3 &\leq 0.4(x_1 + x_2 + x_3 + x_4) \Rightarrow 0.6x_3 - 0.4x_1 - 0.4x_2 - 0.4x_4 \leq 0, \\ x_4 &\leq 0.4(x_1 + x_2 + x_3 + x_4) \Rightarrow 0.6x_4 - 0.4x_1 - 0.4x_2 - 0.4x_3 \leq 0. \end{aligned}$$

4

a) [13 marks]



b) [4 marks] If change of use permission is obtained and they choose Flats, they gain $8 \times 0.7 - 3 \times 0.3 = 4.7$ million pounds. Similarly with Houses they gain $5 \times 0.7 + 1 \times 0.3 = 3.8$. Thus Flats is best, assuming permission is obtained. If the investment is made the expected profit is thus $0.6 \times 4.7 + 0.4 \times -5 = 0.82 > 0$. Thus the company should invest, and go for Flats if change of use permission is obtained.

c) [4 marks] Denote the probability of strong demand by p . Then if permission is obtained and they choose Flats, they gain $8 \times p - 3 \times (1 - p) = 11p - 3$ million pounds. Similarly with Houses they gain $5 \times p + 1 \times (1 - p) = 1 + 4p$. Thus Flats is best if $11p - 3 > 1 + 4p \Rightarrow p > 4/7$, assuming permission is obtained. If the investment is made the expected profit is thus $0.6 \times \max(11p - 3, 1 + 4p) + 0.4 \times -5 = \max(6.6p - 3.8, 2.4p - 1.4)$. The equality point $4/7$ yields a negative profit, and any p less than this means that No Investment is best. Thus we can forget the Houses option, and Flats is best if $6.6p - 3.8 > 0$ i.e. $p > 0.576$, otherwise No Investment is best.

d) [5 marks] Now the probability of permission being granted and strong demand occurring is $0.8 \times 0.7 = 0.56$, and the probability of permission being granted and weak demand occurring is $0.5 \times 0.3 = 0.15$. The probability of permission not being granted is thus $1 - 0.56 - 0.15 = 0.29$. Thus if investment and Flats is chosen the expected profit is $8 \times 0.56 - 3 \times 0.15 - 5 \times 0.29 = 2.58$ and if investment and Houses

is chosen the expected profit is $5 \times 0.56 + 1 \times 0.15 - 5 \times 0.29 = 1.5$. 2.58 is larger than 1.5 and 0, thus investment and Flats should be chosen.

e) [4 marks] We simply replace the monetary returns by their utilities in the above calculations. Thus if change of use permission is obtained the expected utility for Flats is $1 \times 0.7 + 0.25 \times 0.3 = 0.775$ and for Houses is $0.9 \times 0.7 + 0.75 \times 0.3 = 0.855$. Thus Houses is now the best policy. If investment is made the expected utility is $0.6 \times 0.855 + 0.4 \times 0 = 0.513 > 0.5$ which is the utility of 0. Thus the company should invest, and go for Houses if change of use permission is obtained.